Fundamental Groups of Commuting Elements in Lie Groups

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Abstract

We compute the fundamental group of the spaces of ordered commuting n-tuples of elements in the Lie groups $SU(2)$, $U(2)$ and $SO(3)$. For $SO(3)$ the mod-2 cohomology of the components of these spaces is also obtained.

1 Introduction

In this paper we calculate the fundamental groups of the connected components of the spaces

$$M_n(G) := Hom(\mathbb{Z}^n, G), \text{ where } G \text{ is one of } SO(3),\ SU(2) \text{ or } U(2).$$

The space $M_n(G)$ is just the space of ordered commuting n-tuples of elements from G, topologized as a subset of G^n.

The spaces $M_n(SU(2))$ and $M_n(U(2))$ are connected (see [1]), but $M_n(SO(3))$ has many components if $n > 1$. One of the components is the one containing the element $(id, id, \ldots, id)$; see Section 2. The other components are all homeomorphic to $V_2(\mathbb{R}^3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $V_2(\mathbb{R}^3)$ is the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^3$ and the action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $V_2(\mathbb{R}^3)$ is given by

$$(\epsilon_1, \epsilon_2)(v_1, v_2) = (\epsilon_1 v_1, \epsilon_2 v_2), \text{ where } \epsilon_j = \pm 1 \text{ and } (v_1, v_2) \in V_2(\mathbb{R}^3).$$

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Under the homeomorphism $SO(3) \to V_2(\mathbb{R}^3)$ given by $A \mapsto (Ae_1, Ae_2)$ the action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $V_2(\mathbb{R}^3)$ corresponds to the action defined by right multiplication by the elements of the group generated by the transformations

$$(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3),\ (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).$$

The orbit space of this action is homeomorphic to $\mathbb{S}^3/Q_8$, where Q_8 is the quaternion group of order eight.

Then $M_n(SO(3))$ will be a disjoint union of many copies of $\mathbb{S}^3/Q_8$ and the component containing $(id, \ldots, id)$. For brevity let $\vec{1}$ denote the n-tuple $(id, \ldots, id)$.

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Definition 1.1. Let $M_n^+(SO(3))$ be the component of $\vec{1}$ in $M_n(SO(3))$, and let $M_n^-(SO(3))$ be the complement $M_n(SO(3)) - M_n^+(SO(3))$.

Our main result is the following

Theorem 1.2. *For all* $n \geq 1$

$$\begin{array}{rcl} \pi_1(M_n^+(SO(3))) & = & \mathbb{Z}_2^n \\ \pi_1(M_n(SU(2))) & = & 0 \\ \pi_1(M_n(U(2))) & = & \mathbb{Z}^n \end{array}$$

The other components of $M_n(SO(3))$, $n > 1$, all have fundamental group Q_8.

Remark 1.3. To prove this theorem we first prove that $\pi_1(M_n^+(SO(3))) = \mathbb{Z}_2^n$, and then use the following property of spaces of homomorphisms (see [4]). Let Γ be a discrete group, $p : \tilde{G} \to G$ a covering of Lie groups, and C a component of the image of the induced map $p_* : Hom(\Gamma, \tilde{G}) \to Hom(\Gamma, G)$. Then $p_* : p_*^{-1}(C) \to C$ is a regular covering with covering group $Hom(\Gamma, Ker\, p)$. Applying this to the universal coverings $SU(2) \to SO(3)$ and $SU(2) \times \mathbb{R} \to U(2)$ induces coverings

$$\mathbb{Z}_2^n \to M_n(SU(2)) \to M_n^+(SO(3))$$

$$\mathbb{Z}^n \to M_n(SU(2)) \times \mathbb{R}^n \to M_n(U(2))$$

Remark 1.4. The spaces of homomorphisms arise in different contexts (see [5]). In physics for instance, the orbit space $Hom(\mathbb{Z}^n, G)/G$, with G acting by conjugation, is the moduli space of isomorphism classes of flat connections on principal G-bundles over the n-dimensional torus. Note that, if G is connected, the map $\pi_0(Hom(\mathbb{Z}^n, G)) \to \pi_0(Hom(\mathbb{Z}^n, G)/G)$ is a bijection of sets. The study of these spaces arises from questions concerning the understanding of the structure of the components of this moduli space and their number. These questions are part of the study of the quantum field theory of gauge theories over the n-dimensional torus (see [3],[6]).

The organization of this paper is as follows. In Section 2 we study the toplogy of $M_n(SO(3))$ and compute its number of components. In Section 3 we prove Theorem 1.2 and apply this result to mapping spaces. Section 4 treats the cohomology of $M_n^+(SO(3))$. Part of the content of this paper is part of the Doctoral Dissertation of the first author ([7]).

2 The Spaces $M_n(SO(3))$

In this section we describe the topolgy of the spaces $M_n(SO(3))$, $n \geq 2$. If A_1, A_2 are commuting elements from $SO(3)$ then there are 2 possibilities: either A_1, A_2 are rotations about a common axis; or A_1, A_2 are involutions about axes meeting at right angles. The first possibility covers the case where one of A_1, A_2 is the identity since the identity can be considered as a rotation about any axis.

It follows that there are 2 possibilities for an n-tuple $(A_1, \ldots, A_n) \in M_n(SO(3))$:

1. Either $A_1, \ldots, A_n$ are all rotations about a common axis L; or

2. There exists at least one pair i, j such that A_i, A_j are involutions about perpendicular axes. If v_i, v_j are unit vectors representing these axes then all the other A_k must be one of id, A_i, A_j or $A_iA_j = A_jA_i$ (the involution about the cross product $v_i \times v_j$).

It is clear that if $\omega(t) = (A_1(t), \dots, A_n(t))$ is a path in $M_n(SO(3))$ then exactly one of the following 2 possibilities occurs: either the rotations $A_1(t), \dots, A_n(t)$ have a common axis $L(t)$ for all t; or there exists a pair i, j such that $A_i(t), A_j(t)$ are involutions about perpendicular axes for all t. In the second case the pair i, j does not depend on t.

Proposition 2.1. *$M_n^+(SO(3))$ is the space of n-tuples $(A_1, \dots, A_n) \in SO(3)^n$ such that all the A_j have a common axis of rotation.*

Proof. Let $A_1, \dots, A_n$ have a common axis of rotation L. Thus $A_1, \dots, A_n$ are rotations about L by some angles $\theta_1, \dots, \theta_n$. We can change all angles to 0 by a path (while maintaining the common axis). Conversely, if $\omega(t) = (A_1(t), \dots, A_n(t))$ is a path containing $\vec{1}$ then the $A_j(t)$ will have a common axis of rotation for all t (which might change with t). $\square$

Thus any component of $M_n^-(SO(3))$ can be represented by an n-tuple $(A_1, \dots, A_n)$ satisfying possibility 2 above.

Corollary 2.2. *The connected components of $M_2(SO(3))$ are $M_2^{\pm}(SO(3))$.*

Proof. Let (A_1, A_2) be a pair in $M_2^-(SO(3))$. Then there are unit vectors v_1, v_2 in $\mathbb{R}^3$ such that v_1, v_2 are perpendicular and A_1, A_2 are involutions about v_1, v_2 respectively. The pair (v_1, v_2) is not unique since any one of the four pairs $(\pm v_1, \pm v_2)$ will determine the same involutions. In fact there is a 1-1 correspondence between pairs (A_1, A_2) in $M_2^-(SO(3))$ and sets $\{(\pm v_1, \pm v_2)\}$. Thus $M_2^-(SO(3))$ is homeomorphic to the orbit space $V_2(\mathbb{R}^3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $V_2(\mathbb{R}^3)$ is connected so is $M_2^-(SO(3))$. $\square$

Next we determine the number of components of $M_n^-(SO(3))$ for $n > 2$. The following example will give an indication of the complexity.

Example 2.3. *Let (A_1, A_2, A_3) be an element of $M_3^-(SO(3))$. Then there exists at least one pair A_i, A_j without a common axis of rotation. For example suppose A_1, A_2 don't have a common axis. Then A_1, A_2 are involutions about perpendicular axes v_1, v_2, and there are 4 possibilities for $A_3 : A_3 = id, A_1, A_2$ or $A_3 = A_1A_2$. We will see that the triples*

$$(A_1, A_2, id), (A_1, A_2, A_1), (A_1, A_2, A_2), (A_1, A_2, A_1A_2)$$

belong to different components. Similarly if A_1, A_3 or A_2, A_3 don't have a common axis of rotation. This leads to 12 components, but some of them are the same component. An analysis leads to a total of 7 distinct components corresponding to the following 7 triples: (A_1, A_2, id), (A_1, A_2, A_1), (A_1, A_2, A_2), (A_1, A_2, A_1A_2), (A_1, id, A_3), (A_1, A_1, A_3), (id, A_2, A_3); where A_1, A_2 are distinct involutions in the first 4 cases; A_1, A_3 are distinct involutions in the next 2 cases;

and A_2, A_3 are distinct involutions in the last case. These components are all homeomorphic to $\mathbb{S}^3/Q_8$. Thus $M_3(SO(3))$ is homeomorphic to the disjoint union

$$M_3^+(SO(3)) \sqcup \mathbb{S}^3/Q_8 \sqcup \ldots \sqcup \mathbb{S}^3/Q_8,$$

where there are 7 copies of $\mathbb{S}^3/Q_8$.

The pattern of this example holds for all $n \geq 3$. A simple analysis shows that $M_n^-(SO(3))$ consists of n-tuples $\vec{A} = (A_1, \ldots, A_n) \in SO(3)^n$ satisfying the following conditions:

1. Each A_i is either an involution about some axis v_i, or the identity.
2. If A_i, A_j are distinct involutions then their axes are at right angles.
3. There exists at least one pair A_i, A_j of distinct involutions.
4. If A_i, A_j are distinct involutions then every other A_k is one of id, A_i, A_j or A_iA_j.

This leads to 5 possibilities for any element $(B_1, \ldots, B_n) \in M_n^-(SO(3))$:

$$(B_1, B_2, *, \ldots, *), (B_1, id, *, \ldots, *), (id, B_2, *, \ldots, *), (B_1, B_1, *, \ldots, *), (id, id, *, \ldots, *),$$

where B_1, B_2 are distinct involutions about perpendicular axes and the asterisks are choices from amongst $id, B_1, B_2, B_3 = B_1B_2$. The choices must satisfy the conditions above.

These 5 cases account for all components of $M_n^-(SO(3))$, but not all choices lead to distinct components. If $\omega(t) = (B_1(t), B_2(t), \ldots, B_n(t))$ is a path in $M_n^-(SO(3))$ then it is easy to verify the following statements:

1. If some $B_i(0) = id$ then $B_i(t) = id$ for all t.
2. If $B_i(0) = B_j(0)$ then $B_i(t) = B_j(t)$ for all t.
3. If $B_i(0), B_j(0)$ are distinct involutions then so are $B_i(t), B_j(t)$ for all t.
4. If $B_k(0) = B_i(0)B_j(0)$ then $B_k(t) = B_i(t)B_j(t)$ for all t.

These 4 statements are used repeatedly in the proof of the next theorem.

Theorem 2.4. *The number of components of $M_n^-(SO(3))$ is*

$$\begin{cases} \frac{1}{6}(4^n - 3 \times 2^n + 2) & \textit{if } n \textit{ is even} \\[2ex] \frac{2}{3}(4^{n-1} - 1) - 2^{n-1} + 1 & \textit{if } n \textit{ is odd} \end{cases}$$

Moreover, each component is homeomorphic to $\mathbb{S}^3/Q_8$.

Proof. Let x_n denote the number of components. The first 3 values of x_n are $x_1 = 0$, $x_2 = 1$ and $x_3 = 7$, in agreement with the statement in the theorem. We consider the above 5 possibilities one by one. First assume $\vec{B} = (B_1, B_2, *, \ldots, *)$. Then different choices of the asterisks lead to different components. Thus the contribution in this case is 4^{n-2}. Next assume $\vec{B} = (B_1, id, *, \ldots, *)$. Then all choices for the asterisks are admissible, except for those choices involving only id and B_1. This leads to $4^{n-2} - 2^{n-2}$ possibilities. However, changing every occurrence of B_2 to B_3, and B_3 to B_2, keeps us in the same component. Thus the total contribution in this case is $(4^{n-2} - 2^{n-2})/2$. This is the same contribution for cases 3 and 4. Finally, there are x_{n-2} components associated to $\vec{B} = (id, id, *, \ldots, *)$. This leads to the recurrence relation

$$x_n = 4^{n-2} + \frac{3}{2}(4^{n-2} - 2^{n-2}) + x_{n-2}$$

Now we solve this recurrence relation for the x_n.

Given any element $(B_1, \ldots, B_n) \in M_n^-(SO(n))$ we select a pair of involutions, say B_i, B_j, with perpendicular axes v_i, v_j. All the other B_k are determined uniquely by B_i, B_j. Thus the element $(v_i, v_j) \in V_2(\mathbb{R}^3)$ determines $(B_1, \ldots, B_n)$. But all the elements $(\pm v_i, \pm v_j)$ also determine $(B_1, \ldots, B_n)$. Thus the component to which $(B_1, \ldots, B_n)$ belongs is homeomorphic to $V_2(\mathbb{R}^3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \mathbb{S}^3/Q_8$. $\square$

3 Fundamental Group of $M_n(G)$

In this section we prove Theorem 1.2, and we start by finding an appropriate description of $M_n^+(SO(3))$. Let $T^n = (\mathbb{S}^1)^n$ denote the n-torus. Then

Theorem 3.1. *$M_n^+(SO(3))$ is homeomorphic to the quotient space $\mathbb{S}^2 \times T^n/\sim$, where $\sim$ is the equivalence relation generated by*

$$(v, z_1, \ldots, z_n) \sim (-v, \bar{z}_1, \ldots, \bar{z}_n) \quad \textit{and} \quad (v, \vec{1}) \sim (v', \vec{1}) \textit{ for all } v, v' \in \mathbb{S}^2, z_i \in \mathbb{S}^1.$$

Proof. If $(A_1, \ldots, A_n) \in M_n^+(SO(3))$ then there exists $v \in \mathbb{S}^2$ such that $A_1, \ldots, A_n$ are rotations about v. Let $z_j \in \mathbb{S}^1$ be the elements corresponding to these rotations. The $(n+1)$-tuple $(v, z_1, \ldots, z_n)$ is not unique. For example, if one of the A_i's is not the identity then $(-v, \bar{z}_1, \ldots, \bar{z}_n)$ determines the same n-tuple of rotations. On the other hand, if all the A_i's are the identity then any element $v \in \mathbb{S}^2$ is an axis of rotation. $\square$

We will use the notation $[v, z_1, \ldots, z_n]$ to denote the equivalence class of $(v, z_1, \ldots, z_n)$. Thus $x_0 = [v, \vec{1}] \in \mathbb{S}^2 \times T^n/\sim$ is a single point, which we choose to be the base point. It corresponds to the n-tuple $(id, \ldots, id) \in M_n^+(SO(3))$. Then $M_n^+(SO(3))$ is locally homeomorphic to $\mathbb{R}^{n+2}$ everywhere except at the point x_0 where it is singular.

Proof of Theorem 1.2: Notice that the result holds for $n = 1$ since $Hom(\mathbb{Z}, G)$ is homeomorphic to G. The first step is to compute $\pi_1(M_n^+(SO(3)))$. Let $T_0^n = T^n - \{\vec{1}\}$ and $M_n^+ =$

$M_n^+(SO(3))$. Removing the singular point $x_0 = [v, \vec{1}]$ from M_n^+ we have $M_n^+ - \{x_0\} \cong \mathbb{S}^2 \times T_0^n/\mathbb{Z}_2$, see Theorem 3.1. If t denotes the generator of $\mathbb{Z}_2$ then the $\mathbb{Z}_2$ action on $\mathbb{S}^2 \times T_0^n$ is given by

$$t(v, z_1, \ldots, z_n) = (-v, \bar{z}_1, \ldots, \bar{z}_n),\ v \in \mathbb{S}^2, z_j \in \mathbb{S}^1$$

This action is fixed point free and so there is a two-fold covering $\mathbb{S}^2 \times T_0^n \xrightarrow{p} M_n^+ - \{x_0\}$ and a short exact sequence

$$1 \to \pi_1(\mathbb{S}^2 \times T_0^n) \to \pi_1(M_n^+ - \{x_0\}) \to \mathbb{Z}_2 \to 1$$

Let $\mathbf{n}$ denote the north pole of $\mathbb{S}^2$. Then for base points in $\mathbb{S}^2 \times T_0^n$ and $M_n^+ - \{x_0\}$ we take $(\mathbf{n}, -1, \ldots, -1) = (\mathbf{n}, -\vec{1})$ and $[\mathbf{n}, -1, \ldots, -1] = [\mathbf{n}, -\vec{1}]$ respectively.

This sequence splits. To see this note that the composite $\mathbb{S}^2 \to \mathbb{S}^2 \times T_0^n \to \mathbb{S}^2$ is the identity, where the first map is $v \mapsto (v, -\vec{1})$ and the second is just the projection. Both maps are equivariant with respect to the $\mathbb{Z}_2$-actions, and therefore $M_n^+ - \{x_0\}$ retracts onto $\mathbb{R}P^2$.

First we consider the case $n = 2$. Choose -1 to be the base point in $\mathbb{S}^1$. The above formula for the action of $\mathbb{Z}_2$ also defines a $\mathbb{Z}_2$ action on $\mathbb{S}^2 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$. This action is fixed point free. The inclusion $\mathbb{S}^2 \times (\mathbb{S}^1 \vee \mathbb{S}^1) \subset \mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ is equivariant and there exists a $\mathbb{Z}_2$-equivariant strong deformation retract from $\mathbb{S}^2 \times T_0^2$ onto $\mathbb{S}^2 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$. Let a_1, a_2 be the generators $(\mathbf{n}, \mathbb{S}^1, -1)$ and $(\mathbf{n}, -1, \mathbb{S}^1)$ of $\pi_1(\mathbb{S}^2 \times T_0^2) = \mathbb{Z} * \mathbb{Z}$. See the Figure below.

The involution $t : \mathbb{S}^2 \times T_0^2 \to \mathbb{S}^2 \times T_0^2$ induces isomorphisms

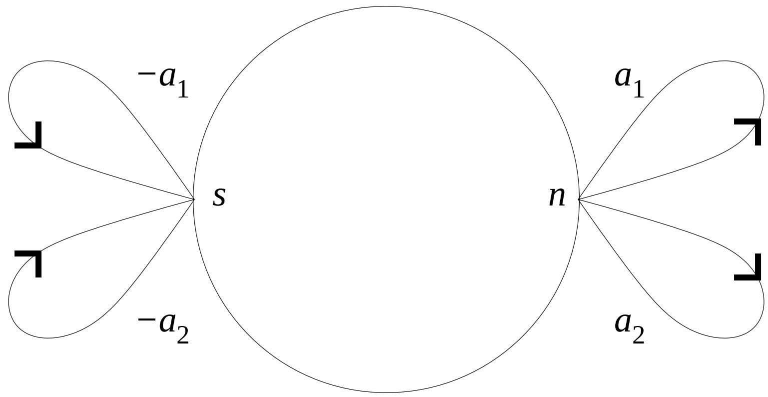
$$\begin{array}{ccc}
\pi_1(\mathbb{S}^2 \times (\mathbb{S}^1 \vee \mathbb{S}^1), \{\mathbf{n}, -1, -1\}) & \xrightarrow{c} & \pi_1(\mathbb{S}^2 \times (\mathbb{S}^1 \vee \mathbb{S}^1), \{\mathbf{s}, -1, -1\}) \\
\pi_1(\mathbb{S}^2 \vee (\mathbb{S}^1 \vee \mathbb{S}^1), \{\mathbf{n}, -1, -1\}) & \xrightarrow{c} & \pi_1(\mathbb{S}^2 \vee (\mathbb{S}^1 \vee \mathbb{S}^1), \{\mathbf{n}, -1, -1\})
\end{array}$$

where $\mathbf{s} = -\mathbf{n}$ is the south pole in $\mathbb{S}^2$.

We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{S}^2 \vee_{\mathbf{n}} (\mathbb{S}^1 \vee \mathbb{S}^1) \arrow[rr, "t"] \arrow[d, "i_{\mathbf{n}}"] & & \mathbb{S}^2 \vee_{\mathbf{s}} (\mathbb{S}^1 \vee \mathbb{S}^1) \arrow[d, "i_{\mathbf{s}}"] \\
\mathbb{S}^2 \times_{\mathbf{n}} (\mathbb{S}^1 \vee \mathbb{S}^1) \arrow[rr, "t"] \arrow[dr, "p"'] & & \mathbb{S}^2 \times_{\mathbf{s}} (\mathbb{S}^1 \vee \mathbb{S}^1) \arrow[dl, "p"] \\
& M_2^+ - \{x_0\} &
\end{tikzcd}$$

where $i_{\mathbf{n}}$ and $i_{\mathbf{s}}$ are inclusions. Here the subscripts $\mathbf{n}$ and $\mathbf{s}$ refer to the north and south poles respectively, which we take to be base points of $\mathbb{S}^2$ in the one point unions. The inclusions $i_{\mathbf{n}}, i_{\mathbf{s}}$ induce isomorphims on π_1 and therefore $p_*\pi_1(\mathbb{S}^2 \vee_{\mathbf{n}} (\mathbb{S}^1 \vee \mathbb{S}^1)) = p_*\pi_1(\mathbb{S}^2 \vee_{\mathbf{s}} (\mathbb{S}^1 \vee \mathbb{S}^1))$. Thus t sends a_1 to the loop based at s but with the opposite orientation (similarly for a_2). See the Figure below.



We now have $\pi_1(M_2^+ - \{x_0\}) =< a_1, a_2, t \mid t^2 = 1, a_1^t = a_1^{-1}, a_2^t = a_2^{-1} >$.

For the computation of $\pi_1(M_n^+ - \{x_0\}),\ n \geq 3$, note that the inclusion $T_0^n \subset T^n$ induces an isomorphism on π_1. Therefore $\pi_1(T_0^n) =< a_1, \ldots, a_n \mid [a_i, a_j] = 1\ \forall\ i, j >$. The various inclusions of T_0^2 into T_0^n (corresponding to pairs of generators) show that the action of t on the generators is still given by $a_i^t = a_i^{-1}$. Thus

$$\pi_1(M_n^+ - x_0) =< a_1, \ldots, a_n, t \mid t^2 = 1, [a_i, a_j] = 1, a_i^t = a_i^{-1} >, \quad \text{for } n \geq 3.$$

The final step in the calculation of $\pi_1(M_n^+)$ is to use van Kampen's theorem. To do this let $U \subset \mathbb{S}^1$ be a small open connected neighbourhood of $1 \in \mathbb{S}^1$ which is invariant under conjugation. Here small means $-1 \notin U$. Then $N_n = \mathbb{S}^2 \times U^n / \sim$ is a contractible neighborhood of x_0 in M_n^+. We apply van Kampen's theorem to the situation $M_n^+ = (M_n^+ - \{x_0\}) \cup N_n$.

The intersection $N_n \cap (M_n^+ - \{x_0\})$ is homotopy equivalent to $(\mathbb{S}^2 \times \mathbb{S}^{n-1})/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by multiplication by -1 on both factors. Therefore $\pi_1(N_n \cap (M_n^+ - \{x_0\})) \cong \mathbb{Z}$ when $n = 2$, and $\mathbb{Z}_2$ when $n \geq 3$. Thus we need to understand the homomorphism induced by the inclusion $N_n \cap (M_n^+ - \{x_0\}) \to M_n^+ - \{x_0\}$.

When $n = 2$ the inclusion of $N_2 \cap (M_2^+ - \{x_0\})$ into $M_2^+ - \{x_0\}$ induces the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "2"] & \mathbb{Z} * \mathbb{Z} \arrow[d] \\
\pi_1(N_2 \cap (M_2^+ - \{x_0\})) \arrow[r] \arrow[d] & \pi_1(M_2^+ - \{x_0\}) \arrow[d] \\
\mathbb{Z}_2 \arrow[r, "="] & \mathbb{Z}_2
\end{tikzcd}$$

where the map on top is the commutator map. So if the generator of $\pi_1(N_2 \cap (M_2^+ - \{x_0\}))$ is sent to $w \in \pi_1(M_2^+ - \{x_0\})$, then $w^2 = [a_1, a_2]$, and the image of w in $\mathbb{Z}_2$ is t. Thus we can write $w = a_1^{n_1} a_2^{m_1} \cdots a_1^{n_r} a_2^{m_r} t$ with $n_i, m_i \in \mathbb{Z}$. Then

$$w^2 = a_1^{n_1} a_2^{m_1} \cdots a_1^{n_r} a_2^{m_r} a_1^{-n_1} a_2^{-m_1} \cdots a_1^{-n_r} a_2^{-m_r} = a_1 a_2 a_1^{-1} a_2^{-1}$$

which occurs only if $r = 1$ and $n_1 = m_1 = 1$. It follows that $w = a_1a_2t$. Thus

$$\pi_1(M_2^+) =< a_1, a_2, t \mid t^2 = 1, a_1^t = a_1^{-1}, a_2^t = a_2^{-1}, a_1a_2t = 1 >$$

and routine computations show that this is the Klein four group.

For $n \geq 3$ the inclusion map $N_n \cap (M_n^+ - \{x_0\}) \to M_n^+ - \{x_0\}$ can be understood by looking at the following diagram

$$\begin{tikzcd}
\mathbb{S}^2 \times \mathbb{S}^1 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{S}^2 \times T_0^2 \arrow[dr] \arrow[dd] & \\
& \mathbb{S}^2 \times S^{n-1} \arrow[rr] \arrow[dd] & & \mathbb{S}^2 \times T_0^n \arrow[dd] \\
N_2 \cap (M_2^+ - \{x_0\}) \arrow[rr] \arrow[dr] & & M_2^+ - \{x_0\} \arrow[dr] & \\
& N_n \cap (M_n^+ - \{x_0\}) \arrow[rr] & & M_n^+ - \{x_0\}
\end{tikzcd}$$

Note that the map $N_2 \cap (M_2^+ - \{x_0\}) \to N_n \cap (M_n^+ - \{x_0\})$ induces the canonical projection $\mathbb{Z} \to \mathbb{Z}_2$. A chase argument shows that the inclusion $N_n \cap (M_n^+ - \{x_0\}) \to M_n^+ - \{x_0\}$ imposes the relation a_1a_2t as well, and therefore

$$\pi_1(M_n^+) =< a_1, \ldots, a_n, t \mid t^2 = 1, [a_i, a_j] = 1, a_i^t = a_i^{-1}, a_1a_2t = 1 > .$$

By performing some routine computations we see that this group is isomorphic to $\mathbb{Z}_2^n$. This completes the proof of Theorem 1.2 for $SO(3)$. The cases of $SU(2)$ and $U(2)$ follow from Remark 1.3. □

Since the map $\pi_1(\vee_n G) \to \pi_1(G^n)$ is an epimorphism, it follows that the inclusion maps

$$M_n^+(G) \to G^n \quad \text{if} \;\; G = SO(3)$$

$$M_n(G) \to G^n \quad \text{if} \;\; G = SU(2), U(2)$$

are isomorphisms in π_1 for all $n \geq 1$. Recall that there is a map $Hom(\Gamma, G) \to Map_*(B\Gamma, BG)$, where $Map_*(B\Gamma, BG)$ is the space of pointed maps from the classifying space of Γ into the classifying space of G. Let $Map_*^+(T^n, BG)$ be the component of the map induced by the trivial representation.

Corollary 3.2. *The maps*

$$M_n^+(G) \to Map_*^+(T^n, BG) \quad \text{if} \;\; G = SO(3)$$

$$M_n(G) \to Map_*^+(T^n, BG) \quad \text{if} \;\; G = U(2)$$

are injective in π_1 for all $n \geq 1$.

Proof. By induction on n, with the case $n = 1$ being trivial. Assume $n > 1$, and note that there is a commutative diagram

$$\begin{tikzcd}
M_n^+(SO(3)) \arrow[r] \arrow[d] & Map_*^+(B\pi_1(T^n), BSO(3)) \arrow[d] \\
Hom(\pi_1(T^{n-1} \vee \mathbb{S}^1), SO(3)) \arrow[r] \arrow[d] & Map_*^+(B\pi_1(T^{n-1} \vee \mathbb{S}^1), BSO(3)) \arrow[d] \\
Hom(\pi_1(T^{n-1}), SO(3)) \times SO(3) \arrow[r] & Map_*^+(B\pi_1(T^{n-1}), BSO(3)) \times SO(3)
\end{tikzcd}$$

in which the bottom map is injective in π_1 by inductive hypothesis, the lower vertical maps are homeomorphisms, and the upper left vertical map is injective in π_1. Thus the map on top is also injective as wanted. The proof for $U(2)$ is the same. $\square$

Remark 3.3. We have the following observations.

1. The two-fold cover $\mathbb{Z}_2 \to \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ allows us to study $Hom(\mathbb{Z}^n, SO(4))$. Let $M_n^+(SO(4))$ be the component covered by $Hom(\mathbb{Z}^n, \mathbb{S}^3 \times \mathbb{S}^3)$. Since $Hom(\mathbb{Z}^n, \mathbb{S}^3 \times \mathbb{S}^3)$ is homeomorphic to $Hom(\mathbb{Z}^n, \mathbb{S}^3) \times Hom(\mathbb{Z}^n, \mathbb{S}^3)$, it follows that

$$\pi_1(M_n^+(SO(4))) = \mathbb{Z}_2^n$$

2. The space $Hom(\mathbb{Z}^2, SO(4))$ has two components. One is $M_2^+(SO(4))$, which is covered by $\partial^{-1}_{SU(2)^2}(1,1)$, and the other is covered by $\partial^{-1}_{SU(2)^2}(-1,-1)$, where $\partial_{SU(2)^2}$ is the commutator map of $SU(2) \times SU(2)$. Recall $\partial^{-1}_{SU(2)}(-1)$ is homeomorphic to $SO(3)$ (see [2]), so $\partial^{-1}_{SU(2)^2}(-1,-1)$ is homeomorphic to $SO(3) \times SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$, where the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts by left diagonal multiplication when it is thought of as the subgroup of $SO(3)$ generated by the transformations $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$ and $(x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3)$.

3. Corollary 3.2 holds similarly for $SO(4)$, and trivially for $SU(2)$.

4 Homological Computations

In this section we compute the $\mathbb{Z}_2$-cohomology of $M_n^+(SO(3))$. The $\mathbb{Z}_2$-cohomology of the other components of $M_n(SO(3))$ is well-known since these are all homeomorphic to $\mathbb{S}^3/Q_8$. To perform the computation we will use the description of $M_n^+(SO(3))$ that we saw in the proof of Theorem 1.2. The ingredients we have to consider are the spectral sequence of the fibration $\mathbb{S}^2 \times T_0^n \to (M_n^+ - \{x_0\}) \to \mathbb{R}P^\infty$ whose E_2 terms is

$$\mathbb{Z}_2[u] \otimes E(v) \otimes E(x_1, \ldots, x_n)/(x_1 \cdots x_n)$$

with $deg(u) = (1,0)$, $deg(v) = (0,2)$ and $deg(x_i) = (0,1)$; and the spectral sequence of the fibration $\mathbb{S}^2 \times \mathbb{S}^{n-1} \to N_n \cap (M_n^+ - \{x_0\}) \to \mathbb{R}P^\infty$ whose E_2-term is

$$\mathbb{Z}_2[u] \otimes E(v) \otimes E(w)$$

with $deg(u) = (1, 0)$, $deg(v) = (0, 2)$ and $deg(w) = (0, n-1)$. Note that in both cases $d_2(v) = u^2$, whereas $d_2(x_i) = 0$ since $H^1(M_n^+ - \{x_0\}, \mathbb{Z}_2) = \mathbb{Z}_2^{n+1}$. Therefore the first spectral sequence collapses at the third term. As $d_n(w) = u^n$ and $d_j(w) = 0$ for $j \neq n$, the second spectral sequence collapses at the third term when $n = 2$ and at the fourth term when $n \geq 3$.

The last step is to use the Mayer-Vietoris long exact sequence of the pair $(M_n^+ - \{x_0\}, N_n)$ which yields the following: for $n = 2, 3$,

$$H^q(M_2^+(SO(3)), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & q = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 3 \\ \mathbb{Z}_2 & q = 4 \\ 0 & q \geq 5 \end{cases}$$

$$H^q(M_3^+(SO(3)), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & q = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 4 \\ \mathbb{Z}_2 & q = 5 \\ 0 & q \geq 6 \end{cases}$$

whereas for $n \geq 4$,

$$H^q(M_n^+(SO(3)), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & q = 0 \\ \mathbb{Z}_2^n & q = 1 \\ \mathbb{Z}_2^{\binom{n}{1} + \binom{n}{2}} & q = 2 \\ \mathbb{Z}_2^{\binom{n}{q-2} + \binom{n}{q-1} + \binom{n}{q}} & 3 \leq q \leq n \\ \mathbb{Z}_2^{\binom{n}{n-1} + 1} & q = n + 1 \\ \mathbb{Z}_2 & q = n + 2 \\ 0 & q \geq n + 3 \end{cases}$$

So the Euler characteristic of $M_n^+(SO(3))$ is given by

$$\chi(M_n^+(SO(3))) = \begin{cases} 0 & n = 2 \textit{ or odd} \\ 2 + n(n-1) - \binom{n}{k-1} - \binom{n}{k} - \binom{n}{k+1} & n = 2k, \quad k \geq 2 \end{cases}$$

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